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Market versus Political Decision Making

Implications from Discounted Utility Maximization

By

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This paper studies the relation between institutional change and socio-economic circumstances within the choice theoretic framework of a two sector model of optimal capital accumulation. Institutional change is seen as reflected in the importance of political decision making. It is shown that the opportunity costs of political decision making might decrease if the social rate of time preference does increase. Furthermore, political decision making is positively (negatively) related to the rate of depreciation of private (public) capital. These results confirm arguments that there might be a positive relation between economic progress and social stability on the one hand and political decision making on the other hand.

I. Introduction

It has been argued, namely by Olson (1982), that economic progress eventually brings about institutional change creating major obstacles to the efficient working of market forces. Thereby, economic progress will slow down and will finally be halted. Neumann (1985) shows that this process is not irreversible. In his model, institutional change is closely related to the social rate of time preference, which, in turn, depends positively on per capita income. Hence, when economic growth slows down time preference will decrease, and market forces will regain social acceptance. The related institutional change will trigger a new upswing.

These arguments depend heavily on the proposed relation between institutional change and the socio-economic circumstances within which the latter takes place. It is the concern of this paper to highlight this relation within a choice theoretic framework.

In order to adopt this perspective it is necessary to make the idea of institutional change operational. I shall do so within two different kinds of models. Consider, first, a general equilibrium type framework where the government acts as if it maximized the utility of a representative citizen. A simplifying picture depicts Western

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industrialized societies as composed of two sectors: the private sector, where market forces determine the supply of private goods, and the public sector, where collective or political decisions govern the supply of public goods. Within this context, the relative size of the public sector indicates the importance of political decision making; its variations reflect institutional change. Thus, an appropriate modelling strategy for the question at hand is to consider market and political allocation as production techniques of private and public goods, respectively.

With respect to the private sector, this approach is fairly well known. It is the procedure of the neoclassical theory of optimal growth to depict a market economy in a one sector model where capital and labour produce private goods. If the public sector is modelled likewise it is essential to give sense to the notions of labour and capital input in public production. Of course, labour and capital input might simply refer to the physical quantity of labour and of capital employed by the government. Yet, in as much as the supply of public goods is not merely a matter of physical production, but of collective decisions, too, the concept of public labour necessarily extends to the human effort involved in public decision making. By the same token, the notion of public capital must comprise both physical capital and the institutional prerequisites for public decision making. Since the constitutional and the political structure of a nation underwent change, public capital is exposed to depreciation like private capital is.

Slightly different notions of public labour and capital pertain to the second kind of model, and in order to emphasize the differences I shall use the adjective 'political' instead of 'public'. Think of a representative household whose utility depends on private goods as well as on her participation in political decision making. This assumption may be justified in either way: there are public goods, where a link between their supply and individually taken actions exists, and/or our agent derives satisfaction from the propagation of her ideological and political beliefs. Institutional change, then, does result from shifts of emphasis put on political instead of market decision making. In this case, political labour refers to her effort spent for various kinds of political action, such as, e.g., voting, lobbying, and founding or joining interest groups. Political capital consists of physical capital (think of an interest group's office) and of the organizational, formal and informal prerequisites for effective political influence (the organizational body of a political party or an interest group, established relations to legislators, the goodwill created by political campaigns, knowledge and information that enable political judgments).

These considerations should suffice to motivate the strategy of the paper. In the next section I shall develop a two-sector model where a representative firm produces a private good and where the government produces a public good. The instantaneous utility of the representative household is a function of both kinds of goods. The government's policy is to maximize the discounted utility of the household. The comparative-statics of the stationary allocation with respect to the model's parameters, namely, the social rate of time preference and the rates of depreciation of private and public capital, provide insights into the relation between the socio-economic environment and the related institutional change. In Section III a few, chiefly formal, modifications transform this model into a partial equilibrium model. A representative citizen allocates labour and capital between private production and political participation. Again, the hypothesis of intertemporal utility maximization underlies the agent's decision making. The comparative-static properties of this model enforce the results obtained from the first model and offer additional insights. An Appendix covers mathematical details. Since the two models differ mathematically only slightly, I only discuss model two there. The main findings of this paper are summarized in the concluding Section IV. They provide support of the thesis that there is a positive relation between the rate of economic progress and social stability on the one hand and the tendency towards increasing political decision making on the other hand.

II. Private and Public Production under Optimal Government Policy

This section considers a two-sector economy with three types of agents: a representative firm, N identical households, and the government. There are two produced goods. The factors of production are labour (n) and capital (k). The division of labour between public and private production may change at each point of time. It is, however, not possible to transfer capital from one sector to the other. The size of net investment alone alters a sector's capital equipment. In the following presentation of the formal model, I suppress the time argument of all variables. A dot denotes differentiation with respect to time. All variables refer to per capita magnitudes. Variables pertaining to the private (public) sector carry the superscript $p\,(g)$.

The firm combines labour, n^p, and capital, k^p, to produce $y^p = f(n^p, k^p)$ units of output. The production function f is linearly homogenous and twice continuously differentiable. The partial derivatives satisfy: $f_i > 0, f_{ii} < 0$, and $f_{ij} > 0$, $i, j = 1, 2$, where the subscript 1 (2) denotes partial differentiation with respect to labour (capital). The firm aims at maximizing its discounted cash flow. Let w denote the real wage rate, i^p gross investment, δ^p the constant rate of depreciation of capital, and ϱ the discount rate. The firm's decision is the solution to the following problem:

$$\text{(II.1)} \quad \max \int_0^\infty [f(n^p, k^p) - wn^p - i^p]\, e^{-\varrho t}\, dt$$

$$\begin{aligned} \text{s.t.}\quad & \dot{k}^p = i^p - \delta k^p \\ & i^p, n^p \geq 0 \\ & k^p(0) = k_0^p \end{aligned}$$

In this optimal control problem, the controls are labour and gross investment. It is well known (see, e.g., Arrow and Kurz (1970) or Kamien and Schwartz (1981) that any solution to this problem must satisfy

$$\text{(II.2a)} \quad f_1(n^p, k^p) = w$$

(II.2b) $f_2(n^p, k^p) = \varrho + \delta^p$

(II.2c) $\dot{k}^p = i^p - \delta^p k^p$

The f_i are homogenous of degree zero, and capital at time t is given. Therefore, equations (II.2.a, b) can be solved for n^p and w to yield

(II.3.a) $n^p = k^p \Phi(\varrho + \delta^p),\ \Phi' > 0$

(II.3.b) $w = \Psi(\varrho + \delta^p),\ \Psi' < 0$

Next, consider a representative household. She supplies one unit of labour and spends her net income on consumption. Her (full employment) income consists of wages, w, and dividend payments, $y^p - wn^p - i^p$. Given the tax rate τ her demand for private goods is

(II.4) $c^p = (1 - \tau)[y^p + (1 - n^p)w - i^p]$

The government hires the unemployed work force,

(II.5) $n^g = 1 - n^p$

and buys i^g units of private goods to enlarge the productive capacity of public capital. The requirement of a balanced budget confines government expenditures to

(II.6) $w(1 - n^p) + i^g = \tau[y^p + (1 - n^p)w - i^p]$

Equations (II.4) through (II.6) imply the goods market clearing condition

(II.7) $y^p = c^p + i^p + i^g$

Public capital and labour produce $c^g = h(n^g, k^g)$ units of public goods. The function h is linear homogenous, twice continuously differentiable with partial derivatives $h_i > 0$, $h_{ii} < 0$, $h_{ij} > 0$, $i, j = 1, 2$. The stock of public capital evolves according to

$$\dot{k}^g = i^g - \delta^g k^g\,,$$

where δ^g denotes the rate of depreciation. The government maximizes the discounted utility of a representative household. Utility at time t is a function $u: R_+^2 \to R_+$ of private and public consumption. If the public good is a pure collective good, the N households will be able to consume each unit supplied jointly. Hence, $u = u(c^p, Nc^g)$. Suppose that u is strictly concave, twice continuously differentiable, and satisfies $u_i > 0$, $u_{ii} < 0$, $u_{ij} > 0$, $i, j = 1, 2$. The optimal government policy solves the following problem:

$$\text{(II.8)} \quad \max \int_0^{\infty} u(c^p, Nc^g) e^{-\varrho t} dt$$

$$\text{s.t.} \quad \dot{k}^p = i^p - \delta^p k^p$$
$$\dot{k}^g = i^g - \delta^g k^g$$
$$c^p + i^p + i^g = k^p f(\Phi(\varrho + \delta^p), 1)$$
$$c^g = h(1 - \mathrm{k}^p \Phi(\varrho + \delta^p), k^g)$$
$$\tau = \frac{i^g + w[1 - k^p \Phi(\varrho + \delta^p)]}{c^p + i^g + w[1 - k^p \Phi(\varrho + \delta^p)]}$$
$$k^p(0) = k_0^p$$
$$k^g(0) = k_0^g$$
$$c^p, c^g, i^p, i^g \geq 0$$

where the controls are c^p, c^g, i^p, and i^g. Once the optimal policy has been found, the government buys i^g units of goods, hires $n^g = 1 - \mathrm{n}^p$ units of labour, and does announce the tax rate. Equation (II.4) and the goods market clearing condition (II.7), then, determine private consumption and investment.

Consider an interior, stationary solution to Problem (II.8). This solution satisfies the following equations:

$$\text{(II.9.a)} \quad f(\Phi(\varrho + \delta^p), 1) - (\varrho + \delta^p) = N \frac{u_2(c^p, Nc^g)}{u_1(c^p, \mathrm{N}c^g)} h_1(1 - k^p \Phi(\varrho + \delta^p), k^g)$$

$$\text{(II.9.b)} \quad \varrho + \delta^g = N \frac{u_2(c^p, Nc^g)}{u_1(c^p, Nc^g)} h_2(1 - k^p \Phi(\varrho + \delta^p), k^g)$$

$$\text{(II.9.c)} \quad c^g = h(1 - k^p \Phi(\varrho + \delta^p), k^g)$$

$$\text{(II.9.d)} \quad k^p f(\Phi(\varrho + \delta^p), 1) = c^p + \delta^p k^p + \delta^g k^g$$

Upon consideration of equations (II.2.a, b) the equations above can be substituted by a more convenient set of six equations in the six unknowns c^p, c^g, n^p, n^g, k^p, and k^g. Observe that multiplying equation (II.9.a) by $(k^p/n^p) = 1/\Phi(\varrho + \delta^p)$ yields $f_1 = N(u_2/u_1)h_1$, since $k^p f(\Phi, 1) = f(n^p, k^p) = f_1 n^p + f_2 n^p$ and $\varrho + \delta^p = f_2$. Add equation (II.2.b) and the labour market clearing condition $n^p + n^g = 1$ to System (II.9) to get

$$\text{(II.10.a)} \quad \varrho + \delta^p = f_1(n^p, k^p)$$

$$\text{(II.10.b)} \quad \varrho + \delta^g = N \frac{u_2(c^p, Nc^g)}{u_1(c^p, Nc^g)} h_1(n^g, \mathrm{k}^g)$$

$$\text{(II.10.c)} \quad f_1(n^p, k^p) = N\frac{u_2(c^p, Nc^g)}{u_1(c^p, Nc^g)} h_2(n^g, k^g)$$

$$\text{(II.10.d)} \quad 0 = f(n^p, k^p) - c^p - \delta^p k^p - \delta^g k^g$$

$$\text{(II.10.e)} \quad 0 = h(\mathrm{n}^g, k^g) - c^g$$

$$\text{(II.10.f)} \quad 1 = n^p + n^g$$

Equation (II.10.a) is a well known condition on (private) capital accumulation. It states that the marginal product of the stationary capital stock must be equal to the user costs of capital, $\varrho + \delta^p$. Equation (II.10.b) is the equivalent condition with respect to public capital. The term $N(u_2/u_1)$, the sum of the households' rates of substitution between private and public consumption, defines the price of public goods in terms of private goods. Therefore, equation (II.10.c) states that the marginal product of labour (in units of private goods) is the same in both sectors. The remaining three equations are market clearing conditions on goods and labour.

The stationary allocation determined by System (II.10) depends on the social rate of time preference, ϱ, and the rates of depreciation of private and public capital, δ^p and δ^g, respectively. These variables may be considered as indicators of the socio-economic climate, as I shall explain in a moment. Hence, the comparative-static properties of the model allow to draw conclusions as to the relation between socio-economic conditions and institutional change. It is convenient to indicate the relative importance of public production and political issues by Nc^g/c^p, i.e., the amount of public goods consumed together with each unit of private goods. This relation is independent of income effects if the instantaneous utility function u is homothetic. A somewhat weaker assumption, namely, that substitution effects dominate counteractive income effects, is, however, sufficient to obtain clear cut answers with respect to Nc^g/c^p. Besides this relation, capital intensity and average labour productivity of private production, k^p/n^p and y^p/n^p, respectively, will be used to describe a stationary allocation.

It is straightforward but rather cumbersome exercise to discover the comparative-static properties of System (II.10). I do not reproduce the calculations here, for this would require too much space. I refer the interested reader to the Appendix. Table 1 presents the results.

They depend, of course, on the assumptions made thus far and on the qualifications just mentioned. A simple mechanism is responsible for the model's outcome: changes of exogenous variables entail a reevaluation of resources that alters the opportunity costs of public production. In more detail, there are the following causalities. If the social rate of time preference increases savings decline raising the relative price of capital. As a consequence, capital intensity and average labour productivity of the private sector decline. If production in the private sector is more capital intensive than public production the opportunity costs of public production decrease. Thus, public production expands. If the user costs of private capital in-

Table 1

Endogenous variables	Exogenous variables		
	ϱ	δ^p	δ^g
Nc^g/c^p	$+ \; if \dfrac{k^p}{n^p} > \dfrac{k^g}{n^g}$ $- \; if \dfrac{k^p}{n^p} < \dfrac{k^g}{n^g}$	$+$	$-$
k^p/n^p	$-$	$-$	0
y^p/n^p	$-$	$-$	0

crease via an increase of the rate of depreciation capital intensity and average labour productivity decline. The private sector's costs of production rise and the related shift in demand shrinks its share in the total production of consumer goods. Similarly, the rate of depreciation of public capital is inversely related to the size of public production. It exerts, however, no effect upon the private sector's capital intensity and labour productivity.

In what sense do these results contribute answers to the problem addressed in this paper? Consider, first, the proposed relation between economic progress, time preference, and public production. In the model, a positive link between time preference and public production depends on public production being more labour intensive than private production. Since public capital is a broadly defined concept it is not self evident whether this condition is satisfied. Hence, the model does not a priori establish a positive link between economic prosperity and public production. It hints, however, at another way by which economic progress might increase public production. It is reasonable to assume that the rate of depreciation of private capital is positively related to the rate of technical and economic progress since a rapid succession of new products and techniques of production shortens the economic life time of capital. Therefore, by increasing user costs of private capital rapid economic progress might decrease the costs of public production and shift attention from market to public decision making. By analogy, the rate of depreciation of public capital is related to the rate of institutional and political change. Hence, if economic and political development bring about social stability, they will favour institutional change towards increasing public decision making.

III. Market and Political Decision Making of a Representative Citizen

The focus of this section is upon a representative citizen whose utility at time t depends on her consumption of private goods, c^p, and her level of political activity, c^g, according to $u = u(c^p, c^g)$. Let $y^p = f(n^p, k^p)$ denote the relation between market income (y^p) and labour (n^p) and capital (k^p) devoted to private production. Similarly, $c^g = h(n^g, k^g)$ relates the level of political activity attained (c^g) to the amount of labour effort (n^g) and capital (k^g) spent for political activities. She earns y^p on the market, pays τy^p taxes, and splits her net income, $(1-\tau)y^p$ between consumption (c^p) and investment in private and political capital, i^p and i^g, respectively. Thus, her budget constraint reads $(1-\tau)y^p \geq c^p + i^p + i^g$. Given her decision, the stocks of capital change according to $\dot{k}^j = i^j - \delta^j k^j$, $j = p, g$. Assume that her total labour effort is confined to 1 and that her decision criterion is dicounted utility. Let $\varrho > 0$ denote her rate of time preference. Finally, let the functions u, f, and h have the same properties as they have in Section II. The choice problem of the representative citizen is:

$$\text{(III.1)} \quad \max \int_0^\infty u(c^p, c^g) e^{-\varrho t} dt$$

$$\begin{aligned}
\text{s.t.} \quad & \dot{k}^p = i^p - \delta^p k^p \\
& \dot{k}^g = i^g - \delta^g k^g \\
& (1-\tau) y^p \geq c^p + i^p + i^g \\
& c^g \leq h(n^g, k^g) \\
& 1 \geq = n^p + n^g \\
& k^p(0) = k_0^p \\
& k^g(0) = k_0^g \\
& 0 \leq c^p, c^g, i^p, i^g, n^p, n^g
\end{aligned}$$

A stationary, interior optimal solution to this problem is determined by the following set of equations:

$$\text{(III.2.a)} \quad \varrho + \delta^p = (1-\tau) f_2(n^p, k^p)$$

$$\text{(III.2.b)} \quad \varrho + \delta^g = \frac{u_2(c^p, c^g)}{u_1(c^p, c^g)} h_2(n^g, k^g)$$

$$\text{(III.2.c)} \quad (1-\tau) f_1(n^p, k^p) = \frac{u_2(c^p, c^g)}{u_1(c^p, c^g)} h_1(n^g, k^g)$$

$$\text{(III.2.d)} \quad 0 = (1-\tau) f(n^p, k^p) - c^p - \delta^p k^p - \delta^g k^g$$

(III.2.e) $$0 = h(n^g, k^g) - c^g$$

(III.2.f) $$1 = n^p + n^g$$

Equation (III.2.a) and equation (III.2.b) are the conditions on capital accumulation. They require equality between user costs and marginal productivity of capital. Equation (III.2.c) ensures that both kinds of decision making yield the same (net) marginal productivity of labour in terms of private goods. The remaining three equations of System (III.2) exclude any waste of resources. In particular, net value added, $(1-\tau)f(\cdot) - \delta^p k^p - \delta^g k^g$, is consumed (Equation (III.2.d)), and the available work hours are spent for earning money and exerting political influence (equation (III.2.f)).

Table 2 shows the comparative-static properties of System (III.2). Since the model is mathematically almost identical with the model of the previous section, it should be no surprise that it confirms the results obtained there.

Table 2

Endogenous variables	Exogenous variables			
	ϱ	δ^p	δ^g	τ
c^g/c^p	$+ \text{ if } \frac{k^p}{n^p} > \frac{k^g}{n^g}$ $- \text{ if } \frac{k^p}{n^p} < \frac{k^g}{ng}$	+	−	+
k^p/n^p	−	−	0	−
y^p/n^p	−	−	0	−

Given the differences between the concept of public capital and the concept of political capital, the assumption of market decision making being more capital intensive than political decision making appears to be more reasonable in the context of this model. After all, for most people political participation is related to gathering and processing information, to voting and bargaining, and to volunteering in political campaigns.

From the reasoning behind the model it is not surprising to find the tax rate positively associated with political decision making: the tax rate raises the costs of market decision making by reducing the net benefits from resources devoted to private production. It might be argued, however, that this relation is somewhat counterintuitive if political decision making gives raise to expanding public production. After all, the tax burdon should induce efforts to shrink not to enlarge the public sector. Yet, I should like to argue, it might be perfectly rational to act as predicted by the model: namely, whenever there is reasonable belief that it will be the fellow citizens bearing the tax burdon associated with government growth. If one accepts that point of view, the model hints at a self enforcing mechanism by which an expanding public sector might encourage demand for public goods.

The inverse relation between depreciation of political capital and political decision making might help explain the seemingly inherent tendency of bureaucratic organisations to further expand: within tight and encrusted institutional bodies there is only little danger of loosing once established channels of influence, and the rules of the bureaucratic game hardly change. Hence, the user costs of political capital are comparatively low, and seeking benefits via political decision making is (relatively) profitable.

IV. Conclusion

This paper is focused on the relation between institutional change and socio-economic circumstances within the choice theoretic framework of discounted utility maximization. Institutional change is seen as reflected in the size of public production as compared to the size of private production. The formal structure of a two sector model relates this indicator to the social rate of time preference and the rates of depreciation of private and public capital. These exogenous variables, in turn, reflect the socio-economic climate. As in Uzawa (1968) and Neumann (1985), my argument is that the social rate of time preference does increase with per capita income. Given that private production is more capital intensive than is public production, this hypothesis implies a positive relation between per capita income and public production. If economic progress and technical change bring about high rates of depreciation of private capital economic progress tends to weaken the forces it rests upon by shifting attention from market to political issues. Finally, in as much as the process of socio-economic development tends to increase institutional and political stability, and, thus, lowers the user costs of political capital, it does encourage political decision making.

I should like to point out that a simple price mechanism is responsible for these results: exogenous changes of user costs entail a reevaluation of resources altering the opportunity costs of political decision making. Hence, the model does capture only one link between institutional change and socio-economic circumstances. Its implications should be seen with this qualification.

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Appendix

The current value Lagrangian of Problem (III.1) is

$$\text{(A.1)}\quad \begin{aligned} L := {} & u(c^p, c^g) + \psi^p[i^p - \delta^p k^p] + \psi^g[i^g - \delta^g k^g] \\ & + \lambda_1[(1-\tau)f(n^p, k^p) - c^p - i^p - i^g] + \lambda_2[h(n^g, k^g) - c^g] \\ & + \lambda_3[1 - n^p - n^g] \end{aligned}$$

ψ^p and ψ^g are costate variables. λ_1, λ_2, and λ_3 are Lagrange multipliers. Obviously, L is concave in state and costate variables.

Assume that u, f, and h are well-behaved. Therefore, any maximum of L with respect to the controls will be an interior solution, i.e., $c^p, c^g, i^p, i^g, n^p, n^g > 0$, the Lagrange multipliers are strictly positive, and all three constraints on the control variables are binding. Consequently, the constraint qualification (see, e.g., Kamien and Schwartz (1981), p. 209) is:

$$\text{rank}\begin{pmatrix} 0 & -1 & 0 & (1-\tau)f_1 & -1 & -1 \\ -1 & 0 & h_1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \end{pmatrix} = 3$$

which is obviously satisfied. The necessary conditions for an interior maximum of Problem (III.1) are:

$$\text{(A.2.a)}\quad \frac{\partial L}{\partial c^p} = u_1(c^p, c^g) - \lambda_1 = 0$$

$$\text{(A.2.b)}\quad \frac{\partial L}{\partial c^g} = u_2(c^p, c^g) - \lambda_2 = 0$$

$$\text{(A.2.c)}\quad \frac{\partial L}{\partial i^p} = \psi^p - \lambda_1 = 0$$

(A.2.d) $$\frac{\partial L}{\partial i^g} = \psi^g - \lambda_1 = 0$$

(A.2.e) $$\frac{\partial L}{\partial n^p} = \lambda_1 (1-\tau) f_1(n^p, k^p) - \lambda_3 = 0$$

(A.2.f) $$\frac{\partial L}{\partial n^g} = \lambda_2 h_1(n^g, k^g) - \lambda_3 = 0$$

(A.2.g) $$-\frac{\partial L}{\partial k^p} = \psi^p \delta^p - \lambda_1 (1-\tau) f_2(n^p, k^p) = \dot{\psi}^p - \varrho \psi^p$$

(A.2.h) $$-\frac{\partial L}{\partial k^g} = \psi^g \delta^g - \lambda_2 h_2(n^g, k^g) = \dot{\psi}^g - \varrho \psi^g$$

The transversality conditions are (see, e. g., Arrow and Kurz (1970), p. 49, Proposition 8)

(A.3.a) $$\lim_{t\to\infty} e^{-\varrho t} \psi^j(t) \geq 0, \quad j = p, g$$

(A.3.b) $$\lim_{t\to\infty} e^{-\varrho t} \psi^j(t) k^j(t) = 0, \quad j = p, g$$

If a stationary equilibrium with finite magnitudes of both k^p and k^g is approached (A.2.a) through (A.2.d) ensure finite shadow prices ψ^p and ψ^g. In this instance, $\varrho > 0$ is necessary and sufficient for (A.3) to be satisfied. This, together with the concavity of L, is sufficient for a maximum of Problem (III.1) to exist (see, e. g., Arrow and Kurz (1970), p. 49, Proposition 8).

From (A.2) the conditions defining a stationary equilibrium are derived by putting $\dot{\psi}^p = \dot{\psi}^g = \dot{k}^p = \dot{k}^g$, ψ^p, $\psi^g > 0$. Conditions (A.2), then, reduce to

(A.4.a) $$\frac{u_2(c^p, c^g)}{u_1(c^p, c^g)} h_2(n^g, k^g) = \varrho + \delta^g$$

(A.4.b) $$\frac{u_2(c^p, c^g)}{u_1(c^p, c^g)} h_1(n^g, k^g) = (1-\tau) f_1(n^p, k^p)$$

(A.4.c) $$(1-\tau) f_2(n^p, k^p) = \varrho + \delta^p$$

Add the binding constraints to equations (A.4) in order to arrive at System (III.2) presented in Section III.

Next, differentiate equations (III.2) considering

(A.5) $(1-\tau)[f_1/h_1] = [u_2/u_1]$ and $-\varrho = \delta^p - (1-\tau)f_2$

from (III.2.c) and (III.2.a), respectively. The result is

$$\begin{pmatrix} -\alpha h_2 & -\beta h_2 & -\gamma\phi h_{22} & 0 & -\gamma\phi h_{21} & 0 \\ -\alpha h_1 & -\beta h_1 & -\gamma\phi h_{12} & \gamma f_{12} & -\gamma\phi h_{11} & \gamma f_{11} \\ 1 & 0 & -h_2 & 0 & -h_1 & 0 \\ 0 & 1 & \delta^g & -\varrho & 0 & -\gamma f_1 \\ 0 & 0 & 0 & \gamma f_{22} & 0 & \gamma f_{21} \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} dc^g \\ dc^p \\ dk^g \\ dk^p \\ dn^g \\ dn^p \end{pmatrix} =$$

$$= \begin{pmatrix} -d\varrho - d\delta^g \\ f_1 d\tau \\ 0 \\ -k^p d\delta - k^g d\delta^g - f d\tau \\ d\varrho + d\delta^p + f_2 d\tau \\ 0 \end{pmatrix}$$

where: $\alpha := [u_1 u_{22} - u_2 u_{12}](u_1)^{-2} < 0$, $\beta := [u_1 u_{21} - u_2 u_{11}](u_1)^{-2} > 0$, $\gamma := (1 - \tau) > 0$, and $\phi := f_1/h_1 > 0$. All derivatives are evaluated at the stationary equilibrium. This system of linear equations allows to calculate the partial derivatives $\partial z/\partial x$, $z \in \{c^p, c^g, n^p\}$, $x \in \{\varrho, \delta^p, \delta^g, \tau\}$. The results can be simplified by recognizing $f_{11}f_{22} - (f_{12})^2 = 0$, $h_{11}h_{22} - (h_{12})^2 = 0$ (due to the linear homogeneity of f and h) and by considering (A.5) as well as (III.2.b). Insert the results into

$$\frac{\partial(z_1/z_2)}{\partial x} = [1/z_2]^2\{z_2(\partial z_1/\partial x) - z_1(\partial z_2/\partial x)\}$$

$z_1 \in \{c^g, k^p\}$, $z_2 \in \{c^p, n^p\}$ to get the expressions looked for.

Applying this procedure yields with respect to the relation between c^g/c^p and the rate of time preference:

(A.6) $$\frac{\partial(c^g/c^p)}{\partial\varrho} = \Delta^{-2}(c^p)^{-2}\{\varrho\gamma\pi_1(\beta c^p + \alpha c^g) - \gamma^2\phi\pi_2\}$$

$$\Delta := \gamma^2\phi\{(h_1h_{22} - h_2h_{12})[f_{22}(\beta\gamma f_1 - \alpha h_1) - \varrho f_{21}] + \\ + f_{22}(h_1h_{21} - h_2h_{11})(\alpha h_2 - \beta\delta^g)\} > 0$$

$$\pi_1 := \phi[h_1(h_1h_{22} - h_2h_{21}) + h_2(h_2h_{11} - h_1h_{12})] + h_2(h_2f_{11} - h_1f_{21}) + \\ + h_1(h_1f_{22} - h_2f_{12}) < 0$$

$$\pi_2 := (h_2c^p + \delta^g c^g)(f_{12}h_{21} - h_{11}f_{22}) + \\ + (h_1c^p + \gamma f_1 c^g)(f_{22}h_{12} - f_{12}h_{22}) - \varrho(f_{21}h_{12} - f_{11}h_{22})c^g$$

The first term in the square brackets of (A.6) is an income effect. The second term is a substitution effect. The income effect does not alter the ratio c^g/c^p if the instantaneous utility function u is homothetic. To prove this assertion calculate the partial derivatives $u_i, u_{ij}, i, j = 1, 2$, from $u := G[c^g g(c^p/c^g)], G' > 0, G'' < 0, g' > 0, g'' < 0$, and insert the results into

$$\beta c^p + \alpha c^g := [(u_1 u_{21} - u_2 u_{11})c^p + (u_1 u_{22} - u_2 u_{12})c^g]\,(u_1)^{-2}$$

to observe that $[\cdot] = 0$.

The substitution effect is positive if the capital intensity of f exceeds the capital intensity of h. Due to the linear homogeneity of f and h,

$$f_{11} = -f_{21}(k^p/n^p)\,, \quad f_{22} = -f_{12}(n^p/k^p)$$

and

$$h_{11} = -h_{21}(k^g/n^g)\,, \quad h_{22} = -h_{12}(n^g/k^g)\,.$$

Therefore, π_2 can be rewritten as

$$\begin{aligned}\pi_2 :=& (h_2 c^p + \delta^g c^g) f_{12} h_{21} [1 - (k^g/n^g)(n^p/k^p)] \\ &- (h_1 c^p + \gamma f_1 c^g) h_{12} f_{12} [(n^p/k^p) - (\mathrm{n}^g/\mathrm{k}^g)] \\ &- \varrho f_{21} h_{12} [1 - (k^p/n^p)(n^g/k^g)] \gtreqless \text{iff} \frac{k^p}{n^p} \gtreqless \frac{k^g}{n^g}\end{aligned}$$

which proves the assertion.

The analytical expressions for the remaining partial derivatives of c^g/c^p are:

$$\begin{aligned}\frac{\partial(c^g/c^p)}{\partial\tau} &= \Delta^{-2}(c^p)^{-2}\gamma^2\phi\{-(f_2 f_{12} - f_1 f_{22})[(h_1 h_{22} - h_2 h_{21})c^p \\ &\quad + (\gamma f_1 h_{22} - \delta^g h_{21})c^g] + \varrho(f_2 f_{11} - f_1 f_{21}) h_{22} c^g\} > 0 \\ \frac{\partial(c^g/c^p)}{\partial\delta^p} &= \Delta^{-2}(c^p)^{-2}\{-\gamma^2\phi[f_{12}(h_1 h_{22} - h_2 h_{21})c^p \\ &\quad + (\varrho h_{22} f_{11} + \delta^g f_{12} h_{21} - \gamma f_1 f_{12} h_{22})c^g]\} > 0 \\ \frac{\partial(c^g/c^p)}{\partial\delta^g} &= \Delta^{-2}(c^p)^{-2}\{-\gamma^2\phi[f_{22}(h_2 h_{11} - h_1 h_{12})c^p \\ &\quad + (\varrho f_{21} h_{12} + \delta^g f_{22} h_{11} - \gamma f_1 f_{22} h_{12})c^g]\} < 0\end{aligned}$$

where all income effects have been eliminated.

The partial derivatives of k^p/n^p are

$$\begin{aligned}\frac{\partial(k^p/n^p)}{\partial\varrho} &= \Delta^{-2}(n^p)^{-2}\gamma\phi\{-\alpha[h_1(h_1 h_{22} - h_2 h_{21}) + h_2(h_2 h_{11} - h_1 h_{12})n^p \\ &\quad + \beta(h_1 h_{22} - h_2 h_{12})(\gamma f_1 n^p + \varrho k^p) + \delta^g \beta(h_2 h_{11} - h_1 h_{21})\} < 0\end{aligned}$$

$$\frac{\partial(k^p/n^p)}{\partial\tau} = \Delta^{-2}(n^p)^{-2}\gamma\phi\{\alpha f_2[h_2(h_2h_{12}-h_1h_{22})+h_2(h_1h_{21}-h_2h_{11})]n^p$$
$$+\beta f_2(h_2h_{12}-h_1h_{22})(-\varrho k^p-\gamma f_1 n^p)+$$
$$+\delta^g\beta f_2(h_2h_{11}-h_1h_{21})n^p\}<0$$

$$\frac{\partial(k^p/n^p}{\partial\delta^p} = \frac{1}{f_2}\,\frac{\partial(k^p/n^p)}{\partial\tau}<0$$

Finally, note that

$$(y^p/n^p) = f[1, \quad (k^p/n^p)]$$

and therefore

$$\frac{\partial(x^p/n^p)}{\partial x} = f_2\,\frac{\partial(k^p/n^p)}{\partial x}$$

which allows to directly calculate the partial derivatives of (y^p/n^p) from those of (k^p/n^p).